Cable Interactions in a Depth Controlled Submersible

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The interaction of a long towing cable with a depth controlled submersible is studied from the point of view of the effect of cable motion on the stability of the vertical motion of the submersible. The planar dynamics of a submerged vehicle towed at constant speed by a long cable are analyzed subject to simplified models of the hydrodynamics. The coupled motions of the vehicle and cable are treated for small perturbations about the towed equilibrium configuration in which the system translates as a whole at the towing speed. In the special case of a straight towed equilibrium state, the modes and frequencies of vibration of the linearized equation governing the cable are derived and the interaction between the vehicle and the massive heavily damped cable is studied. The conditions for stability of the system are determined by the Parameter Plane method whereby the values of two disposable control system parameters which yield stable operation are displayed. It is determined that a relatively small amount of damping opposing the vehicle's motion is adequate to guarantee stability for a broad range of the system gains. The method of analysis presented is applicable to the case of an arbitrary steady state configuration where the modes of vibration about equilibrium are obtained in other than closed form.

Nomenclature

a_k = normalizing coefficient for ψ_k, Eq. (41)
b = $D_v/\rho_c c$, dimensionless drag coefficient
c = $(T_0/\rho_c)^{1/2}$, reference speed
D_v = effective drag coefficient of vehicle, lb-sec/ft
F = tangential force/length, lb/ft
F_k = Laplace transform of f_k
$f_k(\tau)$ = generalized coordinate
J_0, Y_0 = Bessel functions of the first and second kind of order zero
K_1 = position feedback coefficient
K_2 = rate feedback coefficient
$L(t)$ = applied external thrust, lb
$\tilde{L}$ = $L/T_0 \cos\phi_c$
l = cable length, ft
m = $M_A/\rho_c l$, mass ratio
M_A = apparent mass of vehicle, slugs
q = normal component of velocity, ft/sec
Q_k = generalized force
R = normal drag force/length, lb/ft
$\mathbf{R}$ = position of ship (X_I, Y_I)
$\mathbf{r}$ = position vector in (X_I, Y_I)
$\mathbf{r}_e$ = equilibrium position of particle in X,Y
$\mathbf{r}'$ = deviation from equilibrium of particle X,Y
s = arclength along cable, ft or Laplace transform variable
T = tension in cable, lb
t = time, seconds
$\tilde{T}$ = $\bar{T}/T_0$, dimensionless tension
u = tangential component of velocity, ft/sec
U_0 = towship speed, ft/sec
W = "wet" weight of cable/length, lb/ft
x,y = position components
z = $y'/l \cos\phi_c$
z_v = dimensionless displacement of vehicle
$z_v(s)$ = Laplace transform of z_v
α_k = $a_k \times$ (kth Fourier-Bessel coefficient of σ)
β_k = $a_k\mu \times$ (kth Fourier-Bessel coefficient of 1)
δ = $Rl \sin\phi_c \mathcal{C}/T_0 U_0$, dimensionless drag coefficient
λ = $(1-\mu)^{1/2}$
λ_k^2 = eigenvalue of ψ_k
μ = $l(F + W \sin\phi_c)/T_0$
ρ_c = density of cable/length, slug/ft.
σ = s/l, dimensionless length
τ = ct/l, dimensionless time
ϕ = local angle between cable and X_I
ψ_k = eigenfunction of Eq. (35)

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Introduction

The purpose of this investigation is to study the dynamics of an underwater vehicle towed by a cable connected to a surface ship. It is assumed that the mission requires the control of either the depth below the surface or height above the bottom of the underwater vehicle. Attention is focused on the small motions of the cable-vehicle system about the configuration of the towed equilibrium state (TES). Under the usual assumptions regarding the mechanics of the cable and the hydrodynamic forces which act, the nonlinear differential equations governing the motion are derived. These equations are linearized about the TES and the resulting linear partial differential equation for the cable and ordinary differential equation for the vehicle are derived. In the idealized case of a straight equilibrium state (towing at the critical angle) the governing partial differential equation yields a closed form solution involving mode shapes which are expressible in terms of Bessel functions. Coupling between the motions is introduced through the boundary conditions and the effect of the cable vibrations on the response of the vehicle to applied thrust is analyzed. On the assumption that the mission is defined as described above, a feedback law for the applied thrust is introduced and the conditions for stability are determined for the case in which three modes of cable vibration are included.

Kinematic Preliminaries

Figure 1 shows the system under consideration. The system consists of a submerged vehicle towed at a speed U_0 by a long cable of length l attached to a surface ship. The cable is treated as being uniform, inextensible and having no flexural rigidity. The towed body will be characterized by an effective mass M_A and a linear drag coefficient D_v for small vertical velocities. The (X_I, Y_I) coordinate system in Fig. 1 is in-

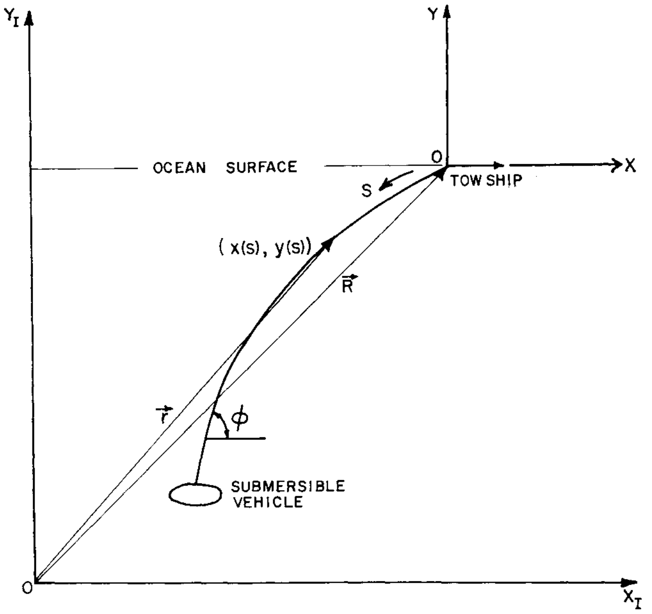

Fig. 1 System schematic.

ertially fixed while the (X,Y) system is fixed to the surface ship and translates with velocity U_0 in the X_I or X direction. The coordinates of a point in the cable are given as functions of arc length s measured from the surface ship (O) as $x(s)$ and $y(s)$. The coordinates of the vehicle are denoted as $[x_v(s), y_v(s)]$. From Fig. 1 the position, velocity, and acceleration of an arbitrary point are related in the two coordinate systems by

$$\begin{aligned} \mathbf{r}(s,t) &= \mathbf{R}(t) + \mathbf{r}^*(s,t) = \mathbf{R}(t) + \mathbf{r}_e(s) + \mathbf{r}'(s,t) \\ \dot{\mathbf{r}}(s,t) &= \mathbf{U}_0 + \dot{\mathbf{r}}'(s,t) = U_0\hat{i} + \dot{\mathbf{r}}'(s,t) \\ \ddot{\mathbf{r}}(s,t) &= \ddot{\mathbf{r}}'(s,t) \end{aligned} \tag{1}$$

where $\mathbf{r}_e(s)$ denotes the position of a point when this system is in the towed equilibrium configuration, i.e., every point translating with velocity U_0. Since the equilibrium configuration of this system may be derived in terms of quadratures when the equations are expressed in local curvilinear coordinates, these will be used for the formulation of the dynamic equations. Figure 2 shows an element of cable with the components of the absolute velocity in both the (X,Y) and a local tangent normal coordinate system. The relationship between tangential and normal components of the absolute

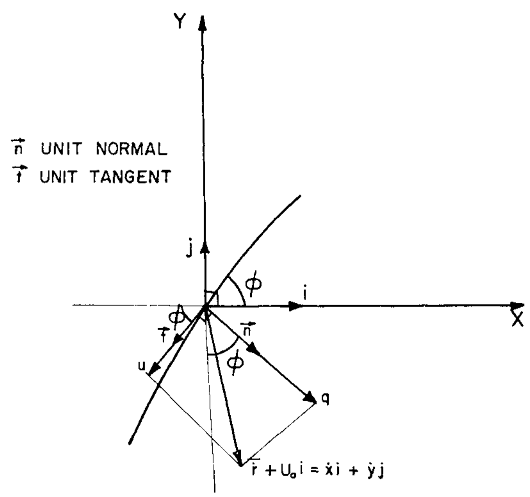

Fig. 2 Coordinate systems.

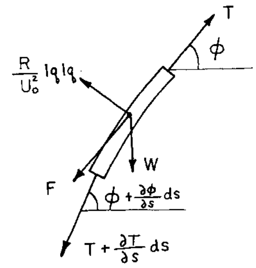

Fig. 3 Free body diagram of cable segment.

velocity u and q, respectively, and the Cartesian components are given by

$$\dot{x} = q\sin\phi - u\cos\phi \qquad \dot{y} = -q\cos\phi - u\sin\phi \tag{2}$$

along with the acceleration relationships

$$\begin{aligned} \ddot{x} &= (\dot{q} + u\dot{\phi})\sin\phi - (\dot{u} - q\dot{\phi})\cos\phi \\ \ddot{y} &= -(\dot{u} - q\dot{\phi})\sin\phi - (\dot{q} + u\dot{\phi})\sin\phi \end{aligned} \tag{3}$$

Equations of Motion

The free body diagram for an infinitesimal element of the cable is given in Fig. 3 in which the normal drag force per unit length is proportional to the square of the normal component of velocity q. The coefficient R is equal to the normal drag on a unit length of cable held perpendicular to a stream at velocity U_0. The force W is the apparent weight per unit length of cable and F is a constant tangential drag force per unit length, while T represents the tension (always >0) in the cable. The dynamic equations for a differential element become†

$$(\partial T/\partial s) + F + W\sin\phi = \rho_c(\dot{u} - q\dot{\phi}) \tag{4}$$

$$(T\partial\phi/\partial s) - (R/U_0^2)|q|q + W\cos\phi = \rho_c(\dot{q} + u\dot{\phi}) \tag{5}$$

The towed equilibrium state (denoted by variables with overbars) corresponds to a translation of the system at velocity U_0 and is governed by

$$(\partial\bar{T}/\partial s) + F + W\sin\bar{\phi} = 0 \tag{6}$$

$$\left(\bar{T}\,\frac{\partial\bar{\phi}}{\partial s}\right) - \left(\frac{R}{U_0^2}\right)U_0^2|\sin\bar{\phi}|\sin\bar{\phi} + W\cos\bar{\phi} = 0 \tag{7}$$

Solutions to Eqs. (6) and (7) have been tabulated by Pode[1] and these solutions will be used as a starting point for modeling of the dynamic case. The dependent variables in this formulation are $T(s)$ and $\phi(s)$ although the solution by quadratures casts ϕ in the role of the independent variable.

Linearization about Towed Equilibrium State

Small motions about the TES are now considered wherein the relevant quantities are represented as the sum of their values in the TES plus a perturbation denoted by a primed variable. Hence, we let

$$T(s,t) = \bar{T}(s) + T'(s,t) \tag{8}$$

$$\phi(s,t) = \bar{\phi}(s) + \phi'(s,t) \tag{9}$$

$$q(s,t) = U_0\sin\bar{\phi}(s) + q'(s,t) \tag{10}$$

$$u(s,t) = -U_0\cos\bar{\phi}(s) + u'(s,t) \tag{11}$$

$$x(s) = \bar{x}(s) + x'(s) \tag{12}$$

$$y(s) = \bar{y}(s) + y'(s) \tag{13}$$

Equations (8–13) are introduced into the equations of motion

† Added mass of cable is ignored.

Eq. (4) and Eq. (5), all products of primed quantities are neglected, and small angle approximations are used: $\cos\phi' \approx 1$ and $\sin\phi' = \phi'$. Similar approximations are used in the acceleration relations Eq. (3) to obtain

$$T'\frac{\partial\bar\phi}{\partial s} + \bar T\frac{\partial\phi'}{\partial s} - 2\frac{R}{U_0^2}\bar q q' - W\sin\bar\phi\phi' = \rho_c(\dot q' + \bar u\dot\phi') = \rho_c(\ddot x'\sin\bar\phi - \ddot y'\cos\bar\phi) \quad (14)$$

where

$$q' = (\dot x'\sin\bar\phi - \dot y'\cos\bar\phi) \quad \text{and} \quad \bar q = U_0\sin\bar\phi$$

and

$$\frac{\partial T'}{\partial s} + W\cos\bar\phi\phi' = -\rho_c(\ddot x'\cos\bar\phi + \ddot y'\sin\bar\phi) \quad (15)$$

Using Eqs. (12) and (13), the relation $\tan\phi = (dy/ds)/(dx/ds)$, and the inextensibility of the cable, an additional kinematic approximation is obtained by neglecting squares of $\partial x'/ds$ and $\partial y'/ds$

$$\phi' = -(1/\cos\bar\phi)(\partial y'/\partial s) \quad (16)$$

With the abbreviations $\cos\bar\phi = \mathcal{C}(s)$, $\sin\bar\phi = \mathcal{S}(s)$, $\tan\bar\phi = \mathfrak{J}(s)$, and introducing Eq. (16) into Eqs. (14) and (15) we obtain

$$(\partial T'/\partial s) - W(\partial y'/\partial s) = -\rho_c[\ddot x'\mathcal{C}(s) + \ddot y'\mathcal{S}(s)] \quad (17)$$

$$T'\frac{\partial\bar\phi}{\partial s} + \bar T(s)\frac{\partial}{\partial s}\left(-\frac{1}{\mathcal{C}(s)}\frac{\partial y'}{\partial s}\right) - \frac{2R}{U_0}\mathcal{S}(s)[\dot x'\mathcal{S}(s) - \dot y'\mathcal{C}(s)] + W\mathfrak{J}(s)\frac{\partial y'}{\partial s} = \rho_c[\ddot x'\mathcal{S}(s) - \ddot y'\mathcal{C}(s)] \quad (18)$$

The condition that the cable is inextensible yields a relationship between $\partial x'/\partial s$ and $\partial y'/\partial s$ which upon neglecting products of small terms becomes

$$\partial x'/\partial s = -\tan\bar\phi(s)(\partial y'/\partial s) \quad (19)$$

The approximations given in Eqs. (16) and (19) which are valid for small perturbations of the configuration essentially specify the perturbed shape in terms of the curvature perturbation, ϕ'. They may be interpreted as stating that under small motions a point of a plane inextensible curve moves along the orthogonal trajectory of the original curve through the point. This relation may be used to express the time derivatives of x' appearing in Eqs. (17) and (18) in terms of y'. From Eq. (19) we have

$$x'(s,t) = x'(0,t) - \int_0^s \mathfrak{J}(s')\frac{\partial y'}{\partial s'}ds' \quad (20)$$

and from the condition that the end of the cable at $s = 0$ is fixed to the towing ship, $x'(0,t) = 0$. Differentiation of Eq. (20) yields

$$\frac{\partial x'}{\partial t} = -\int_0^s \mathfrak{J}(s')\frac{\partial^2 y'}{\partial s'\partial t}ds' \quad (21a)$$

$$\frac{\partial^2 x'}{\partial t^2} = -\int_0^s \mathfrak{J}(s')\frac{\partial^3 y'}{\partial s'\partial t^2}ds' \quad (21b)$$

Introducing Eq. (19) and its consequences into Eqs. (17) and (18) yields a pair of linear partial differential equations in T' y'

$$\frac{\partial T'}{\partial s} - W\frac{\partial y'}{\partial s} = -\rho_c\left[\frac{\partial^2 y'}{\partial t^2}\mathcal{S}(s) - \mathcal{C}(s)\int_0^s \mathfrak{J}(s)\frac{\partial^3 y'}{\partial s'\partial t^2}ds\right] \quad (22)$$

$$T'\frac{\partial\bar\phi}{\partial s} + \bar T(s)\frac{\partial}{\partial s}\left(-\frac{1}{\mathcal{C}(s)}\frac{\partial y'}{\partial s}\right) + \left[\frac{2R}{U_0}\mathcal{S}(s)\right]\times\left[\mathcal{S}(s)\int_0^s\mathfrak{J}(s')\frac{\partial^2 y'}{\partial s'\partial t}ds' - \frac{\partial y'}{\partial t}\mathcal{C}(s)\right] + W\mathfrak{J}(s)\frac{\partial y'}{\partial s} = -\rho_c\left[\mathcal{S}(s)\int_0^s\mathfrak{J}(s')\frac{\partial^3 y'}{\partial s'\partial t^2}ds' - \frac{\partial^2 y'}{\partial t^2}\mathcal{C}(s)\right] \quad (23)$$

Proceeding further Eq. (23) may be solved for T' and the result differentiated with respect to s. This result may be introduced into Eq. (22) to obtain a single equation governing $y'(s,t)$. Thus, in principle, the problem is reduced to the solution of this last mentioned equation. However, the division by $\partial\bar\phi/\partial s$ in Eq. (23) to obtain T' presents serious difficulties whether a numerical or analytical solution is attempted since $\partial\bar\phi/\partial s$ is virtually zero over most of the cable. For this reason, the elimination of T' between Eqs. (22) and (23) is not a practical approach and the analyst is better advised to solve the coupled system governing T' and y' perhaps taking advantage of the smallness of $\partial\bar\phi/\partial s$ in an appropriate perturbation scheme. There is one condition under which in the TES the cable is straight ($\partial\bar\phi/\partial s = 0$). This condition corresponds to the case in which the resultant of the forces applied to the cable as $s = l$ is at an angle $\phi_c + \pi$ where ϕ_c is the critical angle for the cable. The critical angle is defined by setting $\partial\bar\phi/\partial s = 0$ in Eq. (7) which yields

$$W\cos\bar\phi_c = R|\sin\phi_c|\sin\bar\phi_c \quad (24)$$

In what follows for the purposes of the dynamic analysis it will be assumed that the TES is straight which may be viewed either as the neglect of $\partial\phi/\partial s$ in the vicinity of $s = l$ or the assumption that the applied load at $s = l$ is at the angle $\phi_c + \pi$, leaving for further study the case in which $\partial\bar\phi/\partial s$ is retained.‡ This assumption results in considerable simplification of Eq. (22) and Eq. (23) which now appear as

$$\frac{\partial T'}{\partial s} = W\frac{\partial y'}{\partial s} \quad (25)$$

$$-\frac{\bar T(s)}{\mathcal{C}}\frac{\partial^2 y'}{\partial s^2} + \frac{2R}{U_0}\mathfrak{J}\frac{\partial y'}{\partial t} + W\mathfrak{J}\frac{\partial y'}{\partial s} = -\frac{\rho_c}{\mathcal{C}}\frac{\partial^2 y'}{\partial t^2} \quad (26)$$

where the trigonometric functions are now independent of s.

Vibrations about the Equilibrium

Under the assumption $\bar\phi = \phi_c$, $\bar T(s)$ is a linear function of s as may be seen from Eq. (6)

$$\bar T(s) = T_0\left(1 - \frac{\mu s}{l}\right) \quad (27)$$

where

$$T_0 = \bar T(s)|_{s=0}$$

and

$$\mu = \frac{(F + W\sin\phi_c)l}{T_0}$$

With these and the following definitions

$$y'/\mathcal{C}l = z,\ \sigma = s/l,\ \bar T/T_0 = \tilde T,\ \rho_c/T_0 = c^{-2},\ \tau = ct/l$$

‡ It should be recognized that for an arbitrary equilibrium configuration, $\partial\phi/\partial s \neq 0$, there will be a valid linearization for small perturbations which will lead to a system with a self-adjoint spatial operator with variable coefficients. It is envisioned that in these cases the determination of the appropriate mode shapes may be accomplished by any convenient method, and the results employed in an analysis of the type which follows.

we put Eq. (26) and Eq. (27) into the dimensionless forms

$$\tilde{T} = (1 - \mu\sigma) \tag{28}$$

$$\partial/\partial\sigma[(1 - \mu\sigma)(\partial z/\partial\sigma)] - 2\delta\partial z/\partial\tau = \partial^2 z/\partial\tau^2 \tag{29}$$

where $\delta = Rl\mathcal{S}c/T_0U_0$ and we have taken F, the tangential drag force per unit length as zero since it is generally only one or two percent of the normal drag component. The boundary conditions for Eq. (29) are specified in terms of displacements whereby it is required that

$$z(0,\tau) = 0 \quad \text{and} \quad z(1,\tau) = z_v(\tau) \tag{30}$$

where $z_v(\tau)$ is the dimensionless vertical displacement of the submerged vehicle at $s = l$ or $\sigma = 1$. The governing equation Eq. (29) and boundary conditions Eq. (30) are mathematically equivalent to the problem of the vibrations of a heavy stretched string oriented vertically and subjected to large viscous damping. The boundary conditions Eq. (30) correspond to the situation in which the top of the string is fixed and the bottom is subjected to an imposed displacement. As will be shown the normal modes of this system are expressible in terms of the Bessel functions of the first and second kind of order zero, and the natural frequencies are determined by the roots of a frequently encountered transcendental equation.

Normal Modes and Natural Frequencies

The solution of Eq. (29) may be expressed as

$$z(\sigma,\tau) = \bar{z}(\sigma,\tau) + p(\sigma)z_v(\tau) \tag{31}$$

where $p(\sigma)$ is an arbitrary function which will be adjusted so that $\bar{z}$ is subject to homogeneous boundary conditions. This condition may be accomplished by requiring

$$p(0) = 0 \qquad p(1) = 1 \tag{32}$$

Inserting the representation of Eq. (31) into Eq. (29) leads to

$$\frac{\partial}{\partial\sigma}\left[(1 - \mu\sigma)\frac{\partial\bar{z}}{\partial\sigma}\right] - 2\delta\frac{\partial\bar{z}}{\partial\tau} - \frac{\partial^2\bar{z}}{\partial\tau^2} = p(\sigma)\frac{\partial^2 z_v}{\partial\tau^2} + 2\delta p(\sigma)\frac{\partial z_v}{\partial\tau} - z_v(\tau)\frac{d}{d\sigma}\left[(1 - \mu\sigma)\frac{dp}{d\sigma}\right] \tag{33}$$

With the choice, $p(\sigma) = \sigma$, Eq. (32) is satisfied, and $\bar{z}$ admits an expansion in terms of generalized coordinates $f_k(\tau)$ and eigenfunctions or normal modes $\psi_k(\sigma)$

$$\bar{z}(\sigma,\tau) = \sum_{k=1}^{\infty} f_k(\tau)\psi_k(\sigma) \tag{34}$$

The ψ_k must satisfy the following eigenvalue problem

$$\frac{\partial}{\partial\sigma}\left[(1 - \mu\sigma)\frac{\partial\psi_k}{\partial\sigma}\right] + \lambda_k^2\psi_k = 0 \tag{35}$$

$$\psi_k(0) = \psi_k(1) = 0$$

The equation (35) is a form of Bessel's equation[2] and the solution is

$$\psi_k = a_k\left[J_0\left(\frac{2\lambda_k}{\mu}(1 - \mu\sigma)^{1/2}\right) + \gamma_k Y_0\left(\frac{2\lambda_k}{\mu}(1 - \mu\sigma)^{1/2}\right)\right] \tag{36}$$

where we note that since $0 < \mu < 1$ and $0 \leq \sigma \leq 1$ the argument $(2\lambda_k/\mu)(1 - \mu\sigma)^{1/2}$ is bounded away from zero. The eigenvalues λ_k are obtained from the boundary conditions Eq. (35) and Eq. (36) with $\sigma = 0$ and $\sigma = 1$. The governing relation for λ_k is

$$-J_0\left(\frac{2\lambda_k}{\mu}\right)Y_0\left(\frac{2\lambda_k}{\mu}(1 - \mu)^{1/2}\right) + J_0\left(\frac{2\lambda_k}{\mu}(1 - \mu)^{1/2}\right)Y_0\left(\frac{2\lambda_k}{\mu}\right) = 0 \tag{37}$$

The roots of Eq. (37) have been tabulated as a function of λ the ratio of the two arguments which appear in Eq. (37).[3] This ratio is a function of μ only

$$\lambda = (1 - \mu)^{-1/2} \tag{38}$$

and consequently the dimensionless frequencies of the cable vibrating about the TES are functions of T_0 and W. With the normal modes and eigenvalues, λ_k, determined the equation for the generalized coordinates $f_k(\tau)$ follows in the usual manner

$$\ddot{f}_k + 2\delta\dot{f}_k + \lambda_k^2 f_k = Q_k(\tau) \qquad k = 1,2,\ldots\infty \tag{39}$$

$$Q_k(\tau) = \int_0^1 -\psi_k(\sigma)\left[\sigma\left(\frac{\partial^2 z_v}{\partial\tau^2} + 2\delta\frac{\partial z_v}{\partial\tau}\right) + \mu z_v(\tau)\right]d\sigma \tag{40}$$

and the a_k in Eq. (36) have been chosen so that

$$\int_0^1 \psi_k^2 d\sigma = 1 \qquad k = 1,2,\ldots\infty \tag{41}$$

Coupling of the Cable and Vehicle Motions

The interaction between the submerged vehicle and the cable under the excitation of a vertical external force is considered next. The dynamics of the vehicle are characterized by an apparent mass M_A and a linear drag coefficient for motion in the vertical direction D_v. The applied thrust in the vertical direction is denoted by $L(t)$, and the force balance in the y direction becomes

$$L(t) + \bar{T}\cos\bar{\phi}\phi'|_{s=1} = M_A\ddot{y}_v + D_v\dot{y}_v \tag{42}$$

The quantity $\cos\bar{\phi}\phi'$ is related to $\partial y'/\partial s$ by Eq. (16) and Eq. (42) may be rewritten in terms of the dimensionless variables z_v,τ

$$\tilde{L}(\tau) - (1 - \mu)\left.\frac{\partial z}{\partial\sigma}\right|_{\sigma=1} = m\frac{\partial^2 z_v}{\partial\tau^2} + b\frac{\partial z_v}{\partial\tau} \tag{43}$$

where

$$\tilde{L}(\tau) = L(\tau)/T_0\mathcal{C} \qquad M_A/\rho_c l = m \qquad D_v/\rho_c = b \tag{44}$$

The term derived from the cable solution may be evaluated from Eqs. (31) and (34)

$$\partial z/\partial\sigma|_{\sigma=1} = z_v(\tau) + \sum_{k=1}^{\infty} f_k(\tau)\psi_k'(1) \tag{45}$$

Introducing Eq. (45) into Eq. (43) the equation in the time domain governing the coupled dynamics of the vehicle and cable becomes

$$\tilde{L}(\tau) = (1 - \mu)\left[z_v(\tau) + \sum_{k=1}^{\infty} f_k(\tau)\psi_k'(1)\right] = m\frac{\partial^2 z_v}{\partial\tau^2} + b\frac{\partial z_v}{\partial\tau} \tag{46}$$

We note that the $f_k(\tau)$ appearing in Eq. (46) are related to $z_v(\tau)$ by the linear differential equations (41). Thus the governing partial differential equation and the boundary condition at $s = l$ (which actually is of the form of an ordinary differential equation) have been reduced to a system of linear ordinary equations (41) and (46). It may be anticipated that when the external force $\tilde{L}(\tau)$ is related linearly to $z_v(\tau)$

and $\dot{z}_v(\tau)$ the equation of the complete system will resemble that of a closed-loop control system in which the interaction of the cable is manifested by the terms involving the generalized coordinates $f_k(\tau)$.

Closed-Loop Behavior

The cases are considered now in which it is desired to maintain the submerged vehicle at a fixed depth below the surface or at a fixed distance above the bottom. It is hypothesized that sensors capable of detecting the deviation of the vehicle from some reference value of the y coordinate and the rate of change of this deviation are on board the submersible and that a continuous thrust $\tilde{L}(\tau)$ may be exerted as a function of these two signals.§ In terms of the dimensionless variables then we have:

$$\tilde{L}(\tau) = -K_1[z_v(\tau) - z_{\text{ref}}] - K_2[\dot{z}_v(\tau)] \quad (47)$$

where z_{ref} is a reference value of z_v. When the desired path of the submerged vehicle is motion at a constant depth which corresponds to the depth in the TES, $z_{\text{ref}} = 0$. When the objective is bottom-following at a given height above the ocean floor, the reference signal becomes a time varying input which is a function of the bottom profile and the towing speed. For the purposes of a stability analysis in either case we are concerned with the transient response of this system with zero input, and hence we study the closed-loop form of Eq. (46) with $\tilde{L}(\tau)$ given by Eq. (47) and with $z_{\text{ref}} = 0$.

$$m\frac{\partial^2 z_v}{\partial \tau^2} + b\frac{\partial z_v}{\partial \tau} + K_2\frac{\partial z_v}{\partial \tau} + (K_1 + 1 - \mu)z_v + (1-\mu)\sum_{k=1}^{\infty} f_k(\tau)\psi_k'(1) = 0 \quad (48)$$

Taking the Laplace transform of Eqs. (39), (41), and (48) with zero initial conditions permits us to derive the characteristic equation for the system which provides the basis for the stability analysis. Denoting the Laplace transforms of $f_k(\tau)$ and $z_v(\tau)$ by $F_k(s)$ and $z_v(s)$, respectively, we obtain

$$(s^2 + 2\delta s + \lambda_k^2)F_k(s) = Q_k(s) \quad (49)$$

$$\begin{aligned} Q_k(s) &= \int_0^1 \psi_k(\sigma)[\sigma(s^2+2\delta s) + \mu]d\sigma z_v(s) \\ &= [\alpha_k(s^2+2\delta s) + \beta_k]z_v(s) \end{aligned} \quad (50)$$

$$[ms^2 + (b+K_2)s + (K_1 + 1 - \mu)]z_v(s) + (1-\mu)\sum_{k=1}^{\infty} F_k(s)\psi_k'(1) = 0 \quad (51)$$

Combining Eq. (49) and Eq. (50) we obtain

$$F_k(s) = \frac{\alpha_k(s^2+2\delta s) + \beta_k}{s^2 + 2\delta s + \lambda_k^2} z_v(s) \quad (52)$$

which when introduced into Eq. (51) yields the governing equation for the system

$$[ms^2 + (b+K_2)s + (K_1 + 1 - \mu)]z_v(s) + (1-\mu)\sum_{k=1}^{\infty}\frac{\psi_k'(1)(\alpha_k(s^2+2\delta s)+\beta_k)}{s^2+2\delta s+\lambda_k^2} z_v(s) = 0 \quad (53)$$

§ The block diagram for the closed loop system equivalent to this model is shown in Fig. 4. It should be noted that as an example of a controlled flexible system this model does not display the coupling of the elastic response into the sensing function as is usually present in problems in the control of flexible spacecraft.[4] This result is due to the assumption that the position sensors are rigidly mounted on the vehicle and as a consequence the analysis is somewhat simplified relative to the spacecraft case.

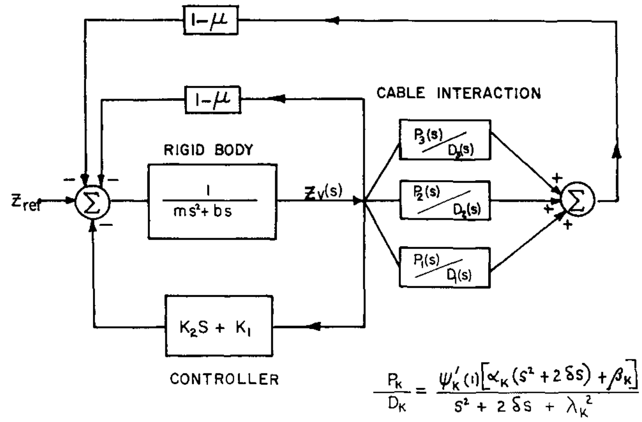

Fig. 4 Block diagram of closed-loop system.

The characteristic equation then becomes

$$ms^2 + (b+k_2)s + (k_1 + 1 - \mu) + (1-\mu)\sum_{k=1}^{\infty}\frac{\psi_k'(1)(\alpha_k(s^2+2\delta s)+\beta_k)}{s^2+2\delta s+\lambda_k^2} = 0 \quad (54)$$

The characteristic equation is transcendental and for practical purposes the sum in Eq. (54) should be truncated at a small number of terms N. If the sum is so truncated then the order of the resulting characteristic polynomial becomes $2 + 2N$. It is clear from Eq. (54) that in the absence of the elastic interactions, the system is stable, and in physical terms the possibility of unstable behavior arises when the vehicle is required to "shake" the massive heavily damped cable. We expect intuitively, that only the first few modes of the cable will be appreciably excited by the low frequency motion of the end. Accordingly, three modes of vibration will be retained in the analysis of an example system.

Stability of Coupled System

If the sum in Eq. (54) is truncated at some finite value of $K = N$ which is equivalent to ignoring higher order modes then Eq. (54) may be rewritten as

$$s^2 + \bar{K}_2 s + \bar{K}_1 + \sum_{k=1}^{N}\frac{h_k}{s^2+2\delta s+\lambda_k^2} = 0 \quad (55)$$

where

$$\bar{K}_2 = (b+K_2)/m$$

$$\bar{K}_1 = \frac{1}{m}\left[K_1 + (1-\mu) + (1-\mu)\sum_{k=1}^{N}\psi_k'(1)\alpha_k\right] \quad (56)$$

$$h_k = \frac{1}{m}[(1-\mu)\psi_k'(1)(\beta_k - \alpha_k\lambda_k^2)]$$

Several alternatives present themselves for determining the conditions which guarantee that the roots of Eq. (56) are confined to the left half of the complex plane in which $s = \sigma + j\omega$. We elect to use the Parameter-Plane method[5] in which we map the left half of the s-plane into a region of the $\bar{K}_1,\bar{K}_2$ plane hence defining all the values of $\bar{K}_1,\bar{K}_2$ which yield stable roots. The technique is applied below to an example in which three modes of cable vibration are retained, i.e., $N = 3$ in Eq. (55).

Example System

The physical characteristics of the example to be studied are given in Table 1. With the values given in Table 1 the coefficients appearing in Eq. (55) become $\lambda_1^2 = 4.52$,

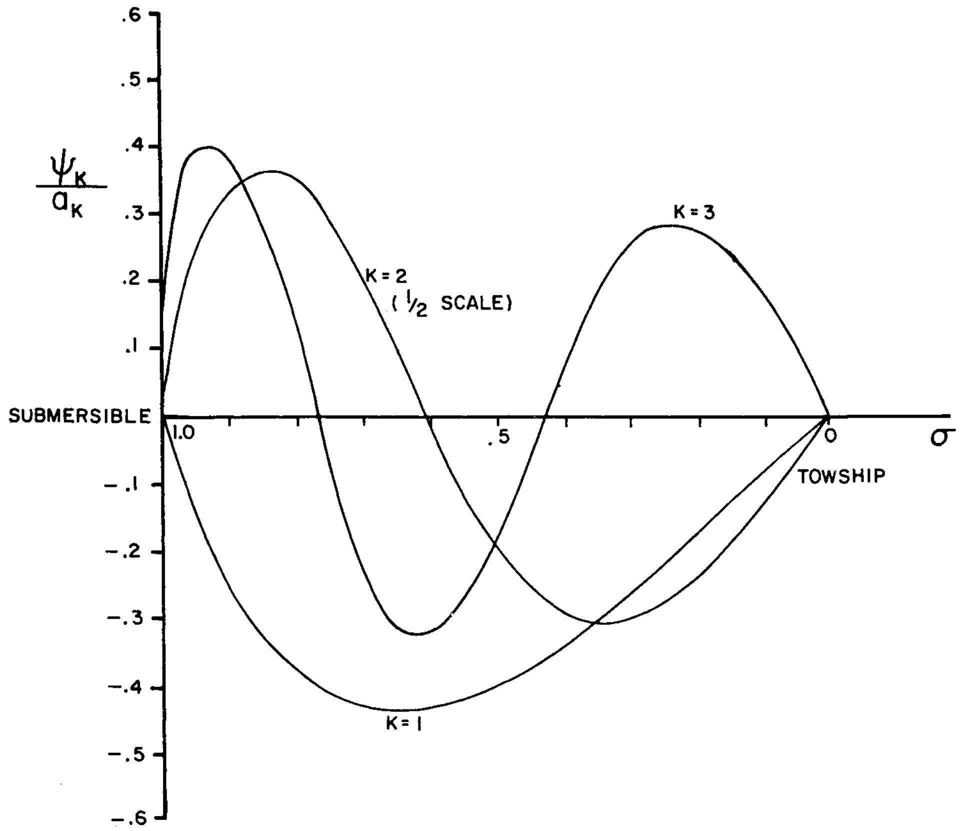

Fig. 5 Mode shapes of cable.

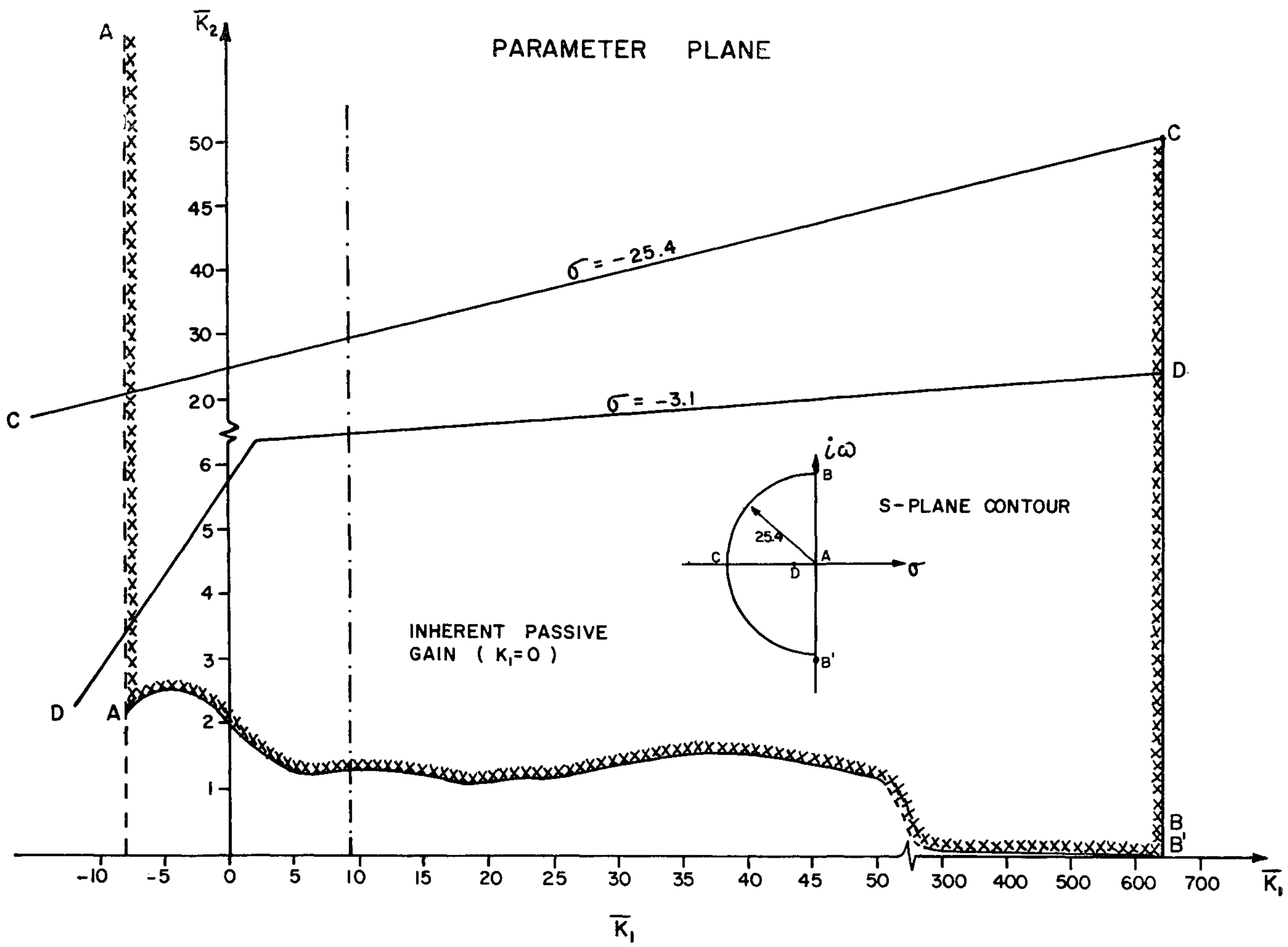

Fig. 6 Stability boundary in the parameter plane $\bar{K}_1$, $\bar{K}_2$.

$h_1 = 15.86$, $\lambda_2^2 = 18.33$, $h_2 = 27.63$, $\lambda_3^2 = 41.37$, $h_3 = 125.8$, and

$$\bar{K}_1 = \frac{1}{m}\left[K_1 + 0.16\left(1 + \sum_{k=1}^{3} \psi_k'(1)\alpha_k\right)\right]$$

The resulting mode shapes are shown for this example in Fig. 5.

Parameter Plane Analysis

The introduction of $s = i\omega$ into Eq. (55) yields an algebraic equation with complex entries. When the real and imaginary parts are equated to zero we obtain

$$\bar{K}_1 = \omega^2 - \sum_{k=1}^{3} \frac{h_k(\lambda_k^2 - \omega^2)}{(\lambda_k^2 - \omega^2)^2 + 4\delta^2\omega^2} \tag{57}$$

$$\bar{K}_2 = 2\delta \sum_{k=1}^{3} \frac{h_k}{(\lambda_k^2 - \omega^2) + 4\delta^2\omega^2} \tag{58}$$

These expressions are a parametric representation of $\bar{K}_1$ and $\bar{K}_2$ as functions of ω. If we allow ω to run from a large negative value to a large positive value then a mapping of the axis $s = i\omega$ into the $\bar{K}_1,\bar{K}_2$ plane is generated. The result of this mapping is shown in Fig. 6. The contour in the left half of the s-plane is closed along a semicircle of radius $|s| = 25.46$ as shown in the inset of Fig. 6. It may be seen that the portion of the ω axis B′AB maps into the similarly labeled curve in the $\bar{K}_1,\bar{K}_2$ plane, with the map of AB and AB′ coinciding. The mapping of the origin in the s-plane must be obtained from Eq. (55) with $s = 0$. It is readily seen from Eq. (55) that with $s = 0$, $\bar{K}_2$ becomes arbitrary and $\bar{K}_1 = 8.06$. Thus, the point A maps into the vertical line AA in the parameter plane. Continuing, the circular arcs BC and B′C which have radius $|s| = 25.46$ map into the nearly vertical curve in the vicinity of $\bar{K}_1 = 640$. The region of stability is shown as that inside the cross-lined boundary in Fig. 6. Some points on the $s = \sigma$ axis which map into straight lines are also shown for reference. It should be noted that the pendulum-like nature of the problem yields an inherent position dependent feedback such that even without the active position feedback, ($K_1 = 0$) there is an inherent residual restoring term. For the example chosen the value of $\bar{K}_1$ with $K_1 = 0$ is 9.5. It may be seen then that if the inherent hydrodynamic damping were sufficient to make $\bar{K}_2 > 1.3$ the system would be stable on a passive basis. Of course, improvements in speed of response, for example, motivate the introduction of active control. In general, a value of $\bar{K}_2 = 1$ appears to be the minimum value which will yield stable operation over a range of $\bar{K}_1$. It can be noted that in the absence of the cable interaction the necessary conditions for stability would simply be $\bar{K}_1 > 0$, $\bar{K}_2 > 0$. This result focuses attention on the linearized hydrodynamic drag in the vertical direction for motions about the towed equilibrium state. If the configuration under study does not yield a satisfactory value of $\bar{K}_2$ on a passive basis, then active rate dependent vertical thrust becomes essential.

Conclusions

The linearization of the governing equations for a towed submerged body with depth control leads to a problem in which the motion of the towing cable may be analyzed in terms of normal modes of vibration of a cable with variable properties. The introduction of the lower modes of vibration of the towing cable into the system transfer function leads to modified stability criteria for the depth control system. The parameter plane analysis of stability conditions reveals that for the example treated the inherent passive restoring force may be adequate to ensure stability if the hydrodynamic drag in the vertical direction exceeds the value which makes $\bar{K}_2 = 1$. If that condition is not met, then the requirement that the submerged vehicle "shake" the more massive long towing cable will lead to instability. The method of analysis presented is applicable to the case of an arbitrary equilibrium configuration where the effect of cable curvature near the submerged vehicle leads to a more difficult eigenvalue problem for the mode shapes.

Table 1 Physical data

Cable length	l	10,000 ft
Towing speed	U_0	3.5 ft/sec
Tension at ship	T_0	7500 lb
Tension at vehicle	T_v	1200 lb
Cable density	W	0.87 lb/ft
Cable drag parameter	R	1.1 lb/ft
μ	μ	0.84
Critical angle	ϕ_c	45.6°
Vehicle mass	M_v	40 slugs
Apparent mass	M_A	53.3 slugs

Appendix: Eigenfunction Relations

The eigenfunctions or mode shapes satisfy Eq. (35) and are given by Eq. (36). The eigenvalues are determined by Eq. (37). To produce modes with unit norm the coefficients a_k are given by

$$a_k^2\left[\left(\frac{1}{\mu} - \sigma\right)\left(J_1\left[\frac{2\lambda_k}{\mu}(1-\mu\sigma)^{1/2}\right] + \gamma_k Y_1\left[\frac{2\lambda_k}{\mu}(1-\mu\sigma)^{1/2}\right]\right)^2\right]_{\sigma=1}^{\sigma=0} = 1 \tag{A1}$$

The coefficients α_k,β_k appearing in the generalized forces may be obtained by straightforward manipulation of the governing differential equation, Eq. (35).

$$\alpha_k = +\frac{1}{\lambda_k^2}\left[(1-\mu\sigma)\frac{\partial\psi_k}{\partial\sigma}\left(\sigma + \frac{\mu}{\lambda_k^2}\right)\right]_{\sigma=1}^{\sigma=0} \tag{A2}$$

$$\beta_k = +\frac{\mu}{\lambda_k^2}\left[(1-\mu\sigma)\frac{\partial\psi_k}{\partial\sigma}\right]_{\sigma=1}^{\sigma=0} \tag{A3}$$

The coefficients γ_k are determined from the eigenvalue equation Eq. (37) along with Eq. (36)

$$\lambda_k = \frac{-J_0\left[\frac{2\lambda_k}{\mu}(1-\mu)^{1/2}\right]}{Y_0\left[\frac{2\lambda_k}{\mu}(1-\mu)^{1/2}\right]} = \frac{-J_0\left(\frac{2\lambda_k}{\mu}\right)}{Y_0\left(\frac{2\lambda_k}{\mu}\right)} \tag{A4}$$

References

[1] Pode, L., "Tables for Computing the Equilibrium Configuration of a Flexible Cable in a Uniform Stream," March 1951, David Taylor Model Basin, Washington, D.C.

[2] Hildebrand, F. B., *Advanced Calculus of Applications*, Prentice-Hall, Englewood Cliffs, N.J., 1962.

[3] Abramowitz, M. and Stegun, I., *Handbook of Mathematical Functions*, Dover, New York, 1965.

[4] Gevarter, W. B., "Basic Relations for Control of Flexible Vehicles," *AIAA Journal*, Vol. 8, No. 4, April 1970, pp. 666–672.

[5] Eisenberg, L., "Stability of Linear Systems with Transport Lag," *IEEE Transactions on Automatic Control*, Vol. AC-11, No. 2, April 1966, pp. 247–254.